On the controllability of laminated beams with Venttsel-type boundary conditions

George José Bautista Sánchez
Instituto de Matemática e Estatística
Universidade Federal Fluminense (UFF), Brasil
geojbs25@gmail.com

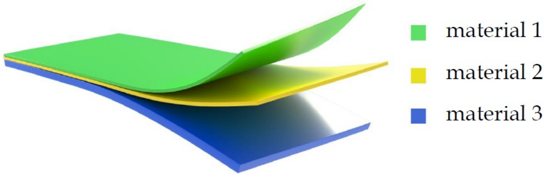
Em colaboração com
Juan Límaco - Universidade Federal Fluminense
Roberto de A. Capistrano–Filho- Universidade Federal de Pernambuco

SEMINARIO EDP IME-UFF

Instituto de Matemática e Estatística
Universidade Federal Fluminense

Maio/2026

Table of contents


material 1
material 2
material 3

The laminated beam model

The laminated beam model was derived from Timoshenko beam theory [11] by S. Hansen and R. Spies [4]. It is a model of two coupled Timoshenko beams given by:

$$\begin{aligned} \rho w_{tt} + G(\psi - w_x)_x &= 0 \quad \text{in} \quad (0, L) \times \mathbb{R}^+, \\ I_\rho \left(3S_{tt} - \psi_{tt}\right) - D(3S_{xx} - \psi_{xx}) - G(\psi - w_x) &= 0 \quad \text{in} \quad (0, L) \times \mathbb{R}^+, \\ 3I_\rho S_{tt} - 3DS_{xx} + 3G(\psi - w_x) + 4\delta_0 S + 4\gamma_0 S_t &= 0 \quad \text{in} \quad (0, L) \times \mathbb{R}^+, \end{aligned} \tag{1}$$

where L represents the length of the beams, and the subscripts x and t represent the partial derivatives with respect to the spatial and temporal variables, respectively. The model describes the dynamics of a two-layer beam bonded together by an intermediate adhesive layer of negligible mass and thickness, which, therefore, does not contribute to the system's kinetic energy. According to the model, the adhesive layer produces a restoring force proportional to the amount of slip.

S. W. Hansen and R. Spies, *Structural damping in a laminated beams due to interfacial slip*, *J. Sound Vibration*, **204**, 183–202, (1997). DOI: 10.1006/jsvi.1996.0913.

The first two equations in (1) come from Timoshenko's theory [11], while the third describes the interaction dynamics between the two layers. It contains a term representing internal frictional damping, commonly known as structural damping.
In (1), $w(x,t)$ and $\psi(x,t)$ represent the transverse displacement and the angle of rotation, respectively. In addition, $S(x,t)$ is proportional to the amount of slip along the interface at time t and the longitudinal spatial variable x. The positive parameters ρ, G, I_ρ, D, δ_0, and γ_0 represent the density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness, and the adhesive structural damping parameter, respectively.
If we change the variables $s=-3S$, $\xi = 3S-\psi$, $\rho_1=\rho$, $\rho_2=I_\rho$, $k=G$, $b=D$, $3\gamma = 4\delta_0$, $3\beta=4\gamma_0$, then the system (1) becomes

$$\begin{aligned} \rho_1 w_{tt} - k\left(w_x+\xi+s\right)_x &= 0 \ \text{ in } (0,L)\times\mathbb{R}^+,\\ \rho_2\xi_{tt} - b\xi_{xx} + k\left(w_x+\xi+s\right) &= 0 \ \text{ in } (0,L)\times\mathbb{R}^+,\\ \rho_2 s_{tt} - bs_{xx} + 3k\left(w_x+\xi+s\right)+\gamma s+\beta s_t &= 0 \ \text{ in } (0,L)\times\mathbb{R}^+. \end{aligned} \tag{2}$$

Timoshenko, S., *On the correction for shear of the differential equation for transverse vibrations of prismatic bars*, *Philosophical Magazine*, **41**, 744–746, (1921). DOI: 10.1080/14786442108636264.

Stability results

In 2005, Wang et al. proved the exponential stability for system (2) with boundary conditions

$$\begin{cases} w(0,t)=\xi(0,t)=s(0,t)=0, & t>0 \\ k_1 w_t(L,t)+w_x(L,t)+\xi(L,t)+s(L,t)=0, & t>0 \\ k_2\xi_t(L,t)+\xi_x(L,t)=0, \ \ s_x(L,t)=0, & t>0, \end{cases} \tag{3}$$

Wang, SM., Xu, GQ., Yung, SP., *Exponential Stabilization of Laminated Beams with Structural Damping and Boundary Feedback Controls*, *SIAM J. Control Optim.*, **44**(5), 1575–1597, (2005). DOI: 10.1137/040610003.

On the other hand, if frictional dissipations are considered in the system (2) (i. e., $\mu_1 w_t$ and $\mu_2 \xi_t$ in $(2)_1$ and $(2)_2$, respectively), Raposo proves the exponential decay of the system.

Raposo, C. A., *Exponential stability for a structure with interfacial slip and frictional damping*, *Appl. Math. Lett.*, **53**, 85–91, (2016). DOI: 10.1016/j.aml.2015.10.005.

Let us consider the laminated beam system with Kelvin-Voigt damping (i. e., $-\mu_1 w_{xxt}$, $-\mu_2 \xi_{xxt}$ and $-\mu_3 s_{xxt}$ in $(2)_1$, $(2)_2$ and $(2)_3$, respectively), Ramos et al., proved that the fully damped system is analytic, while if $\mu_1 = 0$, the partially damped system does not have exponential stability.

Ramos, A. J. A., Freitas, M. M., Cabanillas, V. R., Dos Santos, M. J., & Raposo, C. A., *Stability Results for a Laminated Beam with Kelvin–Voigt Damping*, *Bulletin of the Malaysian Mathematical Sciences Society*, **46**, 152, (2023). DOI: 10.1007/s40840-023-01550-x.

On the whole space $\mathbb{R}$ under the effect of Kelvin-Voigt dampings, we obtain exponential and polynomial decay estimates for the solution of the system and its higher-order derivatives. The polynomial decay rates obtained depend on the regularity of the initial data and vary according to the position of the damping terms.

G.J. Bautista, V.R. Cabanillas, L. Potenciano-Machado, T. Quispe Méndez *Decay rates of strongly damped infinite laminated beams* , *J. Math. Anal. Appl*, **536**, 128229, (2024). DOI: 10.1016/j.jmaa.2024.128229.

So, in context, we are interested in the control properties of the Laminated beam models (2) subject to a class of homogeneous dynamic Venttsel[1] type boundary conditions and prescribed initial data, namely

$$\begin{cases} \rho_1 w_{tt} - k\left(w_x + \xi + s\right)_x = 0, & x \in (0, L), \quad t > 0 \\ \rho_2 \xi_{tt} - b\xi_{xx} + k\left(w_x + \xi + s\right) = 0, & x \in (0, L), \quad t > 0 \\ \rho_2 s_{tt} - bs_{xx} + 3k\left(w_x + \xi + s\right) + \gamma s = 0, & x \in (0, L), \quad t > 0 \\ w(0, t) = \xi(0, t) = s(0, t) = 0, & t > 0 \\ w_{tt}(L, t) + w_x(L, t) + \xi(L, t) + s(L, t) = u_1(t), & t > 0 \\ \xi_{tt}(L, t) + \xi_x(L, t) = u_2(t), & t > 0 \\ s_{tt}(L, t) + s_x(L, t) = u_3(t), & t > 0 \\ (w, \xi, s)(x, 0) = (w_0, \xi_0, s_0)(x), & x \in (0, L) \\ (w_t, \xi_t, s_t)(x, 0) = (w_1, \xi_1, s_1)(x), & x \in (0, L). \end{cases} \tag{4}$$

[1]The name of Alexander Ventcel is often spelled in various ways, such as Wentzell. We refer to the work [12], where this type of boundary condition was first introduced.

To be precise, this work aims to determine whether suitable boundary controls $u_1(t)$, $u_2(t)$, and $u_3(t)$ applied at the boundary of the beam can steer the system's solutions to exhibit prescribed behaviors. This inquiry touches on a central problem in control theory:

Given $T > 0$ *and initial states* $(w_0, w_1, \xi_0, \xi_1, s_0, s_1)$ *within a specified function space, can we determine a control inputs* u_1, u_2 *and* u_3 *such that the system* (4) *admits a solution* (w, ξ, s) *satisfying the initial conditions*

$$\begin{cases} (w, \xi, s)\,(x, 0) = (w_0, \xi_0, s_0)\,(x), & x \in (0, L) \\ (w_t, \xi_t, s_t)\,(x, 0) = (w_1, \xi_1, s_1)\,(x), & x \in (0, L), \end{cases}$$

and the terminal conditions

$$\begin{cases} (w, \xi, s)\,(x, T) = (0, 0, 0), & x \in (0, L) \\ (w_t, \xi_t, s_t)\,(x, T) = (0, 0, 0), & x \in (0, L)? \end{cases}$$

If, given an arbitrary time $T > 0$, one can always find a control inputs to drive the system described by (4) from any given initial state to the equilibrium state, then the system is said to be *null controllable* or *controllable to zero*.

Global well-posedness

In this section, we present the well-posedness results necessary for studying the control system (4). We provide results for both the homogeneous and nonhomogeneous cases.

The homogeneous system

Let us first consider the homogeneous system

$$\begin{cases} \rho_1 w_{tt} - k\left(w_x + \xi + s\right)_x = 0, & x \in (0, L), \quad t > 0, \\ \rho_2 \xi_{tt} - b\xi_{xx} + k\left(w_x + \xi + s\right) = 0, & x \in (0, L), \quad t > 0, \\ \rho_2 s_{tt} - bs_{xx} + 3k\left(w_x + \xi + s\right) + \gamma s = 0, & x \in (0, L), \quad t > 0, \\ w(0, t) = \xi(0, t) = s(0, t) = 0, & t > 0, \\ w_{tt}(L, t) + w_x(L, t) + \xi(L, t) + s(L, t) = 0, & t > 0, \\ \xi_{tt}(L, t) + \xi_x(L, t) = 0, & t > 0, \\ s_{tt}(L, t) + s_x(L, t) = 0, & t > 0, \\ \left(w, \xi, s\right)(x, 0) = \left(w_0, \xi_0, s_0\right)(x), & x \in (0, L), \\ \left(w_t, \xi_t, s_t\right)(x, 0) = \left(w_1, \xi_1, s_1\right)(x), & x \in (0, L). \end{cases} \tag{5}$$

We introduce the new variables

$$\Psi_1 = w_t, \quad \Psi_2 = \xi_t, \quad \Psi_3 = s_t, \quad \Psi_4(\cdot) = \Psi_1(L,\cdot), \quad \Psi_5(\cdot) = \Psi_2(L,\cdot), \quad \Psi_6(\cdot) = \Psi_3(L,\cdot),$$

and define the vector functions

$U = (w, \Psi_1, \xi, \Psi_2, s, \Psi_3, \Psi_4, \Psi_5, \Psi_6)^\top$ and $U_0 = (w_0, w_1, \xi_0, \xi_1, s_0, s_1, \Psi_4(0), \Psi_5(0), \Psi_6(0))^\top$.

Then, the system (5) can be written as an abstract evolution equation

$$\begin{cases} U_t = \mathcal{A}U \\ U(0) = U_0, \end{cases} \tag{6}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U := \begin{bmatrix} \Psi_1 \\ \dfrac{1}{\rho_1}\left[k(w_x + \xi + s)_x\right] \\ \Psi_2 \\ \dfrac{1}{\rho_2}\left[b\xi_{xx} - k(w_x + \xi + s)\right] \\ \Psi_3 \\ \dfrac{1}{\rho_2}\left[bs_{xx} - 3k(w_x + \xi + s) - \gamma s\right] \\ -(w_x(L) + \xi(L) + s(L)) \\ -\xi_x(L) \\ -s_x(L) \end{bmatrix}. \tag{7}$$

To define the phase space $\mathcal{H}$, we consider the Lebesgue space $L^2(0,L)$ with the usual inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We will also use the following Hilbert space

$$H^1_*(0,L)=\left\{w\in H^1(0,L);\, w(0)=0\right\}.$$

Then, we define the phase space as

$$\mathcal{H}=\left[H^1_*(0,L)\times L^2(0,L)\right]^3\times\mathbb{R}^3$$

endowed with the inner product

$$\begin{aligned}\left\langle U,U^\sharp\right\rangle_{\mathcal{H}}=&3\rho_1\left\langle\Psi_1,\Psi_1^\sharp\right\rangle+3\rho_2\left\langle\Psi_2,\Psi_2^\sharp\right\rangle+\rho_2\left\langle\Psi_3,\Psi_3^\sharp\right\rangle+3b\left\langle\xi_x,\xi_x^\sharp\right\rangle\\&+b\left\langle s_x,s_x^\sharp\right\rangle+\gamma\left\langle s,s^\sharp\right\rangle+3k\left\langle w_x+\xi+s,w_x^\sharp+\xi^\sharp+s^\sharp\right\rangle\\&+3k\left(\Psi_4,\Psi_4^\sharp\right)_{\mathbb{R}}+3b\left(\Psi_5,\Psi_5^\sharp\right)_{\mathbb{R}}+b\left(\Psi_6,\Psi_6^\sharp\right)_{\mathbb{R}}\end{aligned}$$

and induced norm

$$\begin{aligned}\|U\|^2_{\mathcal{H}}=&3\rho_1\|\Psi_1\|^2+3\rho_2\|\Psi_2\|^2+\rho_2\|\Psi_3\|^2+3b\|\xi_x\|^2+b\|s_x\|^2+\gamma\|s\|^2+3k\|w_x+\xi+s\|^2\\&+3k|\Psi_4|^2_{\mathbb{R}}+3b|\Psi_5|^2_{\mathbb{R}}+b|\Psi_6|^2_{\mathbb{R}},\end{aligned}$$

for $U=(w,\Psi_1,\xi,\Psi_2,s,\Psi_3,\Psi_4,\Psi_5,\Psi_6)^\top$, $U^\sharp=(w^\sharp,\Psi_1^\sharp,\xi^\sharp,\Psi_2^\sharp,s^\sharp,\Psi_3^\sharp,\Psi_4^\sharp,\Psi_5^\sharp,\Psi_6^\sharp)^\top\in\mathcal{H}$.

So, the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined by (7) has domain given by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (w, \Psi_1, \xi, \Psi_2, s, \Psi_3, \Psi_4, \Psi_5, \Psi_6)^\top \in \mathcal{H};\ (\omega, \xi, s) \in \left[H^2(0,L) \cap H^1_*(0,L)\right]^3, \\ \qquad (\Psi_1, \Psi_2, \Psi_3) \in \left[H^1_*(0,L)\right]^3, \Psi_4 = \Psi_1(L),\ \Psi_5 = \Psi_2(L),\ \Psi_6 = \Psi_3(L), \\ \qquad (w_x(L), \xi_x(L), s_x(L)) \in \mathbb{R}^3. \end{array} \right\}$$

Taking into account the spaces defined in the introduction and denoting by $\rho(\mathcal{A})$ the resolvent set of the operator $\mathcal{A}$. The following result ensures the invertibility of the operator $\mathcal{A}$.

Lemma 2.1

(Bautista, Capistrano–Filho and Límaco, submitted to ZAMM-Zeitschrift fur Angewandte Mathematik und Mechanik)
Let $\mathcal{H}$ and $(\mathcal{A}, D(\mathcal{A}))$ be defined as before. Then, $0 \in \rho(\mathcal{A})$. Moreover, $\mathcal{A}^{-1}$ is compact.

The next result ensures that $\mathcal{A}$ generates a group. The result can be read as follows.

Theorem 2.1

(Bautista, Capistrano–Filho and Límaco, submitted to ZAMM-Zeitschrift fur Angewandte Mathematik und Mechanik)
The operator $(\mathcal{A}, D(\mathcal{A}))$ is an infinitesimal generator of a group of isometries $\{S(t)\}_{t\in\mathbb{R}}$ in $\mathcal{H}$.

As a direct consequence of Lemma 2.1 and Theorem 2.1, and by applying semigroup theory for evolution equations[2], we obtain the following existence and uniqueness result.

Theorem 2.2

(Bautista, Capistrano–Filho and Límaco, submitted to ZAMM-Zeitschrift fur Angewandte Mathematik und Mechanik)
For any $U_0 \in D(\mathcal{A})$, there exists a unique solution U of the system (6) such that

$$U \in C([0,\infty); D(\mathcal{A})) \cap C^1([0,\infty); \mathcal{H}).$$

Moreover, for any $t > 0$, we have that

$$\|U(t)\|^2_{\mathcal{H}} = \|U_0\|^2_{\mathcal{H}}. \tag{8}$$

[2] See, for instance, [9].

The nonhomogeneous system

In this subsection, we focus on the full system (4). We begin with the following result:

Theorem 2.3

(Bautista, Capistrano–Filho and Límaco, submitted to ZAMM-Zeitschrift fur Angewandte Mathematik und Mechanik)
For any $U_0 \in D(\mathcal{A})$ *and* $u_i \in C_0^\infty(0,\infty)$, *for* $i = 1, 2, 3$, *system* (4) *has a unique solution* $U \in C([0,\infty); D(\mathcal{A})) \cap C^1([0,\infty); \mathcal{H})$.

Sketch of proof

Let $\phi_1 \in C^\infty[0, L]$ be a cut-off function, such that $\phi_1(0) = \phi_1(L) = 0$ and $\phi_1'(L) = -1$. If we consider the change of functions

$$\begin{pmatrix} z \\ \eta \\ \varphi \end{pmatrix} = \begin{pmatrix} w \\ \xi \\ s \end{pmatrix} - \begin{pmatrix} \tilde{w} \\ \tilde{\xi} \\ \tilde{s} \end{pmatrix} + \begin{pmatrix} u_1(t)\phi_1(x) \\ u_2(t)\phi_1(x) \\ u_3(t)\phi_1(x) \end{pmatrix}, \tag{9}$$

where $\left(\tilde{w}, \tilde{\xi}, \tilde{s}\right)$ is the unique solution of the system (5) and (z, η, φ) solves the problem

$$\begin{cases} W_t + \mathcal{A}W = \mathcal{K} \\ W(0) = 0, \end{cases}$$

where

$$\begin{cases} W = (z, z_t, \eta, \eta_t, \varphi, \varphi_t, z_t(L,T), \eta_t(L,t), \varphi_t(L,t))^\top, \text{ and} \\ \mathcal{K} = \left(0, \frac{F}{\rho_1}, 0, \frac{G}{\rho_2}, 0, \frac{H}{\rho_2}, 0, 0, 0\right)^\top \in \left[C^1([0,\infty] \times [0,L])\right]^9, \end{cases}$$

with

$$\begin{cases} F(t,x) = & \rho_1 u_1^{(2)}(t)\phi_1(x) - k\left[u_1(t)\phi_1^{(2)}(x) + (u_2(t) + u_3(t))\,\phi_1'(x)\right], \\ G(t,x) = & \rho_2 u_2^{(2)}(t)\phi_2(x) - bu_2(t)\phi_2^{(2)}(x) + k\left[u_1(t)\phi_1'(x) + (u_2(t) + u_3(t))\,\phi_1(x)\right] \\ H(t,x) = & \rho_2 u_3^{(2)}(t)\phi_3(x) - bu_3(t)\phi_3^{(2)}(x) + 3k\left[u_1(t)\phi_1'(x) + (u_2(t) + u_3(t))\,\phi_1(x)\right] \\ & +\gamma u_3(t)\phi_1(x), \end{cases}$$

Since $\mathcal{A}$ generates a group of isometries in $\mathcal{H}$, we have that system above has a unique solution $W \in C([0,\infty);\mathcal{H})$. Then, returning to (9) we conclude the proof. □

Using the previous well-posedness results we will study solutions of the system (4) in the sense of transposition:

Solution by transposition

Definition 2.1

Given $T>0$, $U_0 \in \mathcal{H}'$, $(h_1,h_2,h_3) \in \left(L^2\left(0,T;\left(H^1_*(0,L)\right)^*\right)\right)^3$ and $u_i \in L^2(0,T)$, for $i=1,2,3$, let us consider the non-homogeneous system given by

$$
\begin{cases}
\rho_1 w_{tt} - k\left(w_x+\xi+s\right)_x = h_1, & x\in(0,L),\ t\in(0,T),\\
\rho_2 \xi_{tt} - b\xi_{xx} + k\left(w_x+\xi+s\right) = h_2, & x\in(0,L),\ t\in(0,T),\\
\rho_2 s_{tt} - b s_{xx} + 3k\left(w_x+\xi+s\right) + \gamma s = h_3, & x\in(0,L),\ t\in(0,T),\\
w(0,t)=\xi(0,t)=s(0,t)=0, & t\in(0,T),\\
w_{tt}(L,t)+w_x(L,t)+\xi(L,t)+s(L,t)=u_1(t), & t\in(0,T),\\
\xi_{tt}(L,t)+\xi_x(L,t)=u_2(t), & t\in(0,T),\\
s_{tt}(L,t)+s_x(L,t)=u_3(t), & t\in(0,T),\\
(w,\xi,s)(x,0)=(w_0,\xi_0,s_0)(x), & x\in(0,L)\\
(w_t,\xi_t,s_t)(x,0)=(w_1,\xi_1,s_1)(x), & x\in(0,L).
\end{cases}
\tag{10}
$$

A vector function

$$U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L), \xi_t(L), s_t(L))^\top \in C([0,T]; D(\mathcal{A})),$$

is said to be a solution by transposition to (10) if the following identity holds

$$\left\langle \begin{pmatrix} 3\rho_1 w_t(\cdot,\tau) \\ -3\rho_1 w(\cdot,\tau) \\ 3\rho_2 \xi_t(\cdot,\tau) \\ -3\rho_2 \xi(\cdot,\tau) \\ \rho_2 s_t(\cdot,\tau) \\ -\rho_2 s(\cdot,\tau) \\ -3kw(L,\tau) \\ -3b\xi(L,\tau) \\ -bs(L,\tau) \end{pmatrix}, \begin{pmatrix} \chi_0^\tau \\ \chi_1^\tau \\ \eta_0^\tau \\ \eta_1^\tau \\ \Theta_0^\tau \\ \Theta_1^\tau \\ \chi_t(L,\tau) \\ \eta_t(L,\tau) \\ \Theta_t(L,\tau) \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 3\rho_1 w_1 \\ -3\rho_1 w_0 \\ 3\rho_2 \xi_1 \\ -3\rho_2 \xi_0 \\ \rho_2 s_1 \\ -\rho_2 s_0 \\ -3kw(L,0) \\ -3b\xi(L,0) \\ -bs(L,0) \end{pmatrix}, \begin{pmatrix} \chi(0) \\ \chi_t(0) \\ \eta(0) \\ \eta_t(0) \\ \Theta(0) \\ \Theta_t(0) \\ \chi_t(L,0) \\ \eta_t(L,0) \\ \Theta_t(L,0) \end{pmatrix} \right\rangle \tag{11}$$

$$+ \int_0^\tau \langle (3h_1(t), 3h_2(t, h_3(t))), (\chi(t), \eta(t), \Theta(t)) \rangle_{[(H_*^1(0,L))^*, H_*^1(0,L)]^3}\, dxdt$$

$$+ \Big((3ku_1(t), 3bu_2(t), bu_3(t)), 1_{(0,\tau)}(\chi(L,t), \eta(L,t), \Theta(L,t)) \Big)_{[L^2(0,T)]^3}$$

$$+ ((3kw_t(L,0), 3b\xi_t(L,0), bs_t(L,0), (\chi(L,0), \eta(L,0), \Theta(L,0))))_{\mathbb{R}^3}$$

for any $\tau \in [0, T]$ and $W^\tau \in \mathcal{H}$, with

$$W^\tau = (\chi_0^\tau, \chi_1^\tau, \eta_0^\tau, \eta_1^\tau, \Theta_0^\tau, \Theta_1^\tau, \chi_1^\tau(L), \eta_1^\tau(L), \Theta_1^\tau(L))^\top,$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket in $\mathcal{H}' \times \mathcal{H}$ and $W = (\chi, \chi_t, \eta, \eta_t, \Theta, \Theta_t, \chi_t(L,t), \eta_t(L,t), \Theta_t(L,t))^\top$ is solution of the following adjoint system

$$\begin{cases} \rho_1 \chi_{tt} - k\left(\chi_x + \eta + \Theta\right)_x = 0, & x \in (0,L), \quad t \in (0,\tau), \\ \rho_2 \eta_{tt} - b\eta_{xx} + k\left(\chi_x + \eta + \Theta\right) = 0, & x \in (0,L), \quad t \in (0,\tau), \\ \rho_2 \Theta_{tt} - b\Theta_{xx} + 3k\left(\chi_x + \eta + \Theta\right) + \gamma\Theta = 0, & x \in (0,L), \quad t \in (0,\tau), \\ \chi(0,t) = \eta(0,t) = \Theta(0,t) = 0, & t \in (0,\tau) \\ \chi_{tt}(L,t) + \chi_x(L,t) + \eta(L,t) + \Theta(L,t) = 0, & t \in (0,\tau), \\ \eta_{tt}(L,t) + \Theta_x(L,t) = 0, & t \in (0,\tau), \\ \Theta_{tt}(L,t) + \Theta_x(L,t) = 0, & t \in (0,\tau), \\ (\chi, \eta, \Theta)(x,\tau) = (\chi_0^\tau, \eta_0^\tau, \Theta_0^\tau), & x \in (0,L), \\ (\chi_t, \eta_t, \Theta_t)(x,\tau) = (\chi_1^\tau, \eta_1^\tau, \Theta_1^\tau), & x \in (0,L), \\ (\chi, \eta, \Theta)(L,\tau) = (0,0,0). \end{cases} \tag{12}$$

The next theorem establishes the existence and uniqueness of solutions for the system (4) in the transposition sense.

Theorem 2.4

(Bautista, Capistrano–Filho and Límaco, submitted to ZAMM-Zeitschrift fur Angewandte Mathematik und Mechanik)

Let $T > 0$, $U_0 \in D(\mathcal{A})$, $(h_1, h_2, h_3) \in \left(L^2\left(0, T; \left(H^1_*(0, L)\right)^*\right)\right)^3$ *and* $u_i \in L^2(0, T)$, *for* $i = 1, 2, 3$. *Then, there exists a unique solution*

$$U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L), \xi_t(L), s_t(L))^\top \in C([0, T]; D(\mathcal{A})),$$

of system (10) *which verifies* (11).

Sketch of proof

Let us define a functional Δ given by the right-hand side of (11). Note that Δ is linear. Moreover, it follows that

$$|\Delta\left(W^\tau\right)| \leq C_T\left(\|V_0^1\|_{\mathcal{H}} + \|(u_1, u_2, u_3)\|_{[L^2(0,T)]^3} + \|(h_1, h_2, h_3)\|_{\left(L^2\left(0,T;\left(H^1_*(0,L)\right)^*\right)\right)^3} + \|\left(w_t(L,0), \xi_t(L,0), s_t(L,0)\right)\|_{\mathbb{R}^3}\right)\|W^\tau\|_{\mathcal{H}}.$$

Hence, we obtain that $\Delta \in \mathcal{L}(D(\mathcal{A}); \mathbb{R})$. Thus, from the Riesz representation theorem, we obtain the existence and uniqueness of $U \in D(\mathcal{A})$ satisfying (11). □

Controllability characterization

In this section, we study some boundary controllability properties of the system (4). We start with the following characterization of a control driving system (4) to zero. This kind of result is already classic for dispersive systems (see, for instance, [1] and [8]).

Lemma 3.1

(Bautista, Capistrano–Filho and Límaco, submitted to ZAMM-Zeitschrift fur Angewandte Mathematik und Mechanik)

The initial data $U_0 \in \mathcal{H}'$ is controllable to zero in time $T > 0$ with controls $u_i \in L^2(0,T)$, for $i = 1, 2, 3$, if and only if

$$- \left\langle \begin{pmatrix} 3\rho_1 w_1 \\ -3\rho_1 w_0 \\ 3\rho_2 \xi_1 \\ -3\rho_2 \xi_0 \\ \rho_2 s_1 \\ -\rho_2 s_0 \\ -3kw(L,0) \\ -3b\xi(L,0) \\ -bs(L,0) \end{pmatrix}, \begin{pmatrix} \chi(0) \\ \chi_t(0) \\ \eta(0) \\ \eta_t(0) \\ \Theta(0) \\ \Theta_t(0) \\ \chi_t(L,0) \\ \eta_t(L,0) \\ \Theta_t(L,0) \end{pmatrix} \right\rangle \tag{13}$$

$$- ((3kw_t(L,0), 3b\xi_t(L,0)bs_t(L,0), (\chi(L,0), \eta(L,0), \Theta(L,0))))_{\mathbb{R}^3}$$

$$= 3k \int_0^T u_1(t)\chi(L,t)dt + 3b \int_0^T u_2(t)\eta(L,t)dt + b \int_0^T u_3(t)\Theta(L,t)dt.$$

Here, $\langle \cdot, \cdot \rangle$ is the duality of $\mathcal{H}' \times \mathcal{H}$ and (χ, η, Θ) is any solution of the following adjoint system

$$\begin{cases} \rho_1 \chi_{tt} - k\left(\chi_x + \eta + \Theta\right)_x = 0, & x \in (0, L), \quad t > 0, \\ \rho_2 \eta_{tt} - b\eta_{xx} + k\left(\chi_x + \eta + \Theta\right) = 0, & x \in (0, L), \quad t > 0, \\ \rho_2 \Theta_{tt} - b\Theta_{xx} + 3k\left(\chi_x + \eta + \Theta\right) + \gamma\Theta = 0, & x \in (0, L), \quad t > 0, \\ \chi(0, t) = \eta(0, t) = \Theta(0, t) = 0, & t > 0, \\ \chi_{tt}(L, t) + \chi_x(L, t) + \eta(L, t) + \Theta(L, t) = 0, & t > 0, \\ \eta_{tt}(L, t) + \Theta_x(L, t) = 0, & t > 0, \\ \Theta_{tt}(L, t) + \Theta_x(L, t) = 0, & t > 0, \\ (\chi, \eta, \Theta)(x, T) = \left(\chi_0^T, \eta_0^T, \Theta_0^T\right)(x), & x \in (0, L), \\ (\chi_t, \eta_t, \Theta_t)(x, T) = \left(\chi_1^T, \eta_1^T, \Theta_1^T\right)(x), & x \in (0, L), \end{cases} \tag{14}$$

with $(\chi_0^T, \chi_1^T, \eta_0^T, \eta_1^T, \Theta_0^T, \Theta_1^T, \chi_1^T(L), \eta_1^T(L), \Theta_1^T(L))^\top \in \mathcal{H}$.

Observability inequality

Since we are using the control duality theory of Dolecki and Russell [3] in the set-up of Lions [7], relation (13) may be seen as an optimality condition for the critical points of the functional $\mathcal{J} : \mathcal{H} \to \mathbb{R}$, given by

$$\mathcal{J}(W_0) = \frac{1}{2} \int_0^T \left(|\chi(L,t)|^2 \, dt + |\eta(L,t)|^2 + |\Theta(L,t)|^2 \right) dt + \langle W_0, U_0 \rangle_{\mathcal{H} \times \mathcal{H}'} ,$$

where $W = (\chi, \chi_t, \eta, \eta_t, \Theta, \Theta_t, \chi_t(L,t), \eta_t(L,t), \Theta_t(L,t))^\top$ is solution of the adjoint system (14).

With this in hand, our task is to prove the existence of a minimizer for $\mathcal{J}$. It is well known that the existence of a minimizer to the functional $\mathcal{J}$ follows from the following observability inequality

$$\|W(0)\|^2_{\mathcal{H}} \leq C \int_0^T \left(|\chi(L,t)|^2 \, dt + |\eta(L,t)|^2 + |\Theta(L,t)|^2 \right) dt, \tag{15}$$

for any

$$(\chi_0^T, \chi_1^T, \eta_0^T, \eta_1^T, \Theta_0^T, \Theta_1^T, \chi_1^T(L), \eta_1^T(L), \Theta_1^T(L))^\top \in D(\mathcal{A}^*).$$

Here, $W = (\chi, \chi_t, \eta, \eta_t, \Theta, \Theta_t, \chi_t(L,t), \eta_t(L,t), \Theta_t(L,t))^\top$ is solution of the adjoint system (14).

Now, our efforts are to prove that this inequality holds. Before proving the inequality (15), let us prove intermediary inequalities related to the traces of the solutions of (6). The first one can be read as follows.

Proposition 4.1

(Bautista, Capistrano–Filho and Límaco, submitted to ZAMM-Zeitschrift fur Angewandte Mathematik und Mechanik)
Let $U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L,t), \xi_t(L,t), s_t(L,t))^\top$ the solution of the system (6). Then, for any $T > 0$ and $s \in (\frac{1}{2}, 1)$, there exist a positive constant C such that the following estimate hold

$$\begin{aligned} \|U(t)\|^2_{\mathcal{H}} \leq & C \int_0^T \left(|w_t(L,t)|^2 \, dt + |\xi_t(L,t)|^2 + |s_t(L,t)|^2\right) dt \\ & + C\|(w,\xi,s)\|^2_{L^\infty(0,T;[H^s(0,L)]^3)}. \end{aligned} \tag{16}$$

Sketch of proof

In order to obtain estimate (16), we multiply the first equation in (5) by $3xw_x$, the second equation in (5) by $3x\xi_x$, the third one by xs_x, integrate by parts on $(0,L) \times (0,T)$ and we have that

$$
\begin{aligned}
&\frac{1}{2}\int_0^T \|U(t)\|_{\mathcal{H}}^2 dt \\
&= \frac{1}{2}\int_0^T \Big((3\rho_1 L + 3k)w_t^2(L,t) + (3\rho_2 L + 3b)\xi_t^2(L,t) + (\rho_2 L + b)s_t^2(L,t)\Big)\, dt \\
&- \int_0^L \left[3\rho_1 x w_t w_x + 3\rho_2 x \xi_t \xi_x + \rho_2 x s_t s_x\right]_0^T dx + \frac{bL}{2}\int_0^T \Big(3\xi_x^2(L,t) + s_x^2(L,t)\Big)\, dt \\
&+ 3k\int_0^L\int_0^T (w_x + \xi + s)(\xi + s)\, dt dx + \frac{3kL}{2}\int_0^T (w_x(L,t) + \xi(L,t) + s(L,t))\, w_x(L,t) dt \\
&- \frac{3kL}{2}\int_0^T (w_x(L,t) + \xi(L,t) + s(L,t))(\xi(L,t) + s(L,t))\, dt \\
&- \frac{\gamma L}{2}\int_0^T s^2(L,t) dt + \gamma L\int_0^L\int_0^T s^2 dt dx.
\end{aligned}
\tag{17}
$$

On the other hand, using the Young inequality and from the Sobolev embedding, from (17), we deduce that

$$\begin{aligned}\int_0^T \|U(t)\|^2_{\mathcal{H}} dt \leq & C \int_0^T \left(w_t^2(L,t) + \xi_t^2(L,t) + s_t^2(L,t) \right) dt \\ & + C\|(w,\xi,s)\|^2_{L^\infty(0,T;[H^s(0,L)]^3)},\end{aligned} \tag{18}$$

for any $s \in (\frac{1}{2}, 1)$. Finally, using (8) and the inequality (18), we infer the estimative (16). This finishes the proof. □

Let us now establish the second auxiliary inequality of this section.

Theorem 4.1

(Bautista, Capistrano–Filho and Límaco, submitted to ZAMM-Zeitschrift fur Angewandte Mathematik und Mechanik)
Let $U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L,t), \xi_t(L,t), s_t(L,t))^\top$ *the solution of the system* (6)*. Then, there exists a positive constant* C *such that the following estimate holds*

$$\|U_0\|^2_{\mathcal{H}} \leq C \int_0^T \left(|w_t(L,t)|^2 \, dt + |\xi_t(L,t)|^2 + |s_t(L,t)|^2\right) dt. \tag{19}$$

Sketch of proof

Let us argue by contradiction, following the so-called "compactness-uniqueness" argument[a]. Suppose that (19) does not hold. Thus, there exists a sequence $\{U_0^n\}_{n\in\mathbb{N}} \in \mathcal{H}$, such that

$$\|U^n(t)\|^2_{\mathcal{H}} = \|U_0^n\|^2_{\mathcal{H}} = 1, \quad \text{for all} \quad t \in [0,T], \tag{20}$$

and

$$\int_0^T \left(\left|w_t^n(L,t)\right|^2 dt + \left|\xi_t^n(L,t)\right|^2 + \left|s_t^n(L,t)\right|^2\right) dt \longrightarrow 0, \quad \text{as } n \to \infty, \tag{21}$$

[a] For details of this argument see, for instance, [7, 14].

Since $H^1(0,L) \hookrightarrow H^s(0,L) \hookrightarrow L^2(0,L)$ and the embedding $H^1(0,L) \hookrightarrow H^s(0,L)$ is compact, for all $s \in (\frac{1}{2}, 1)$, from (16), (20) and classical compactness results [10, Corollary 4], we can extract a subsequence of $\{(w^n, \xi^n, s^n)\}_{n\in\mathbb{N}}$, still denoted by $\{(w^n, \xi^n, s^n)\}_{n\in\mathbb{N}}$, such that $\{U^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(0,T;\mathcal{H})$, and we deduce

$$U^n \longrightarrow U \quad \text{strongly in} \quad L^\infty(0,T;\mathcal{H}),$$

$$\|U(t)\|^2_{\mathcal{H}} = 1, \quad \text{for all} \quad t \in [0,T], \tag{22}$$

and

$$0 = \int_0^T \left(|w_t(L,t)|^2 \, dt + |\xi_t(L,t)|^2 + |s_t(L,t)|^2 \right) dt. \tag{23}$$

Thus, we have that the limit $U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L,t), \xi_t(L,t), s_t(L,t))^\top$ satisfies the linear system

$$\begin{cases} \rho_1 w_{tt} - k(w_x + \xi + s)_x = 0, & x \in (0,L), \ t \in (0,T), \\ \rho_2 \xi_{tt} - b\xi_{xx} + k(w_x + \xi + s) = 0, & x \in (0,L), \ t \in (0,T), \\ \rho_2 s_{tt} - bs_{xx} + 3k(w_x + \xi + s) + \gamma s = 0, & x \in (0,L), \ t \in (0,T), \\ w(0,t) = \xi(0,t) = s(0,t) = 0, & t \in (0,T), \\ w_x(L,t) + \xi(L,t) + s(L,t) = 0, & t \in (0,T), \\ \xi_x(L,t) = 0, & t \in (0,T), \\ s_x(L,t) = 0, & t \in (0,T). \end{cases} \tag{24}$$

Let $(\hat{w}, \hat{\xi}, \hat{s})$ the extension by zero of (w, ξ, s), for $x \in (-a,a) \setminus (0,L)$, where $(0,L) \subset (-a,a)$ it is an interval. Then, $(\hat{w}, \hat{\xi}, \hat{s})$ solves, in $D'((0,L) \times (0,T))$, the following system

$$
\begin{cases}
\rho_1 \hat{w}_{tt} - k(\hat{w}_x + \hat{\xi} + \hat{s})_x = 0, & x \in (0, L), \quad t \in (0, T),\\
\rho_2 \hat{\xi}_{tt} - b\hat{\xi}_{xx} + k(\hat{w}_x + \hat{\xi} + \hat{s}) = 0, & x \in (0, L), \quad t \in (0, T),\\
\rho_2 \hat{s}_{tt} - b\hat{s}_{xx} + 3k(\hat{w}_x + \hat{\xi} + \hat{s}) + \gamma \hat{s} = 0, & x \in (0, L), \quad t \in (0, T),\\
\hat{w}(0,t) = \hat{\xi}(0,t) = \hat{s}(0,t) = 0, & t \in (0, T),\\
\hat{w}_x(L,t) + \hat{\xi}(L,t) + \hat{s}(L,t) = 0, & t \in (0, T),\\
\hat{\xi}_x(L,t) = 0, & t \in (0, T),\\
\hat{s}_x(L,t) = 0, & t \in (0, T)\\
(\hat{w}, \hat{\xi}, \hat{s}) = (0,0,0), & x \in (-a, 0] \cup [L, a), \quad t \in (0, T),
\end{cases}
\tag{25}
$$

Then, by Holmgren's uniqueness theorem,

$$(\hat{w}, \hat{\xi}, \hat{s}) \equiv (0,0,0) \text{ in } (-a, a) \times (0, T).$$

Hence, from (24) and (25), we have that $U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L,t), \xi_t(L,t), s_t(L,t))^\top$ satisfies

$$
\begin{cases}
k\left(w_x + \xi + s\right)_x = 0, & x \in (0, L), \quad t \in (0, T),\\
b\xi_{xx} - k\left(w_x + \xi + s\right) = 0, & x \in (0, L), \quad t \in (0, T),\\
bs_{xx} - 3k\left(w_x + \xi + s\right) - \gamma s = 0, & x \in (0, L), \quad t \in (0, T),\\
w(0,t) = \xi(0,t) = s(0,t) = 0, & t \in (0, T),\\
w_x(L,t) + \xi(L,t) + s(L,t) = 0, & t \in (0, T),\\
\xi_x(L,t) = 0, & t \in (0, T),\\
s_x(L,t) = 0, & t \in (0, T).
\end{cases}
\tag{26}
$$

Thanks to the uniqueness of the solution to the previous system established by Lemma 2.1, the only solution to (26) is U = 0. However, this contradicts (22). Therefore, (19) must hold, which completes the proof of the Theorem. □

We are now in a position to prove the observability inequality.

Theorem 4.2

(Bautista, Capistrano–Filho and Límaco, submitted to ZAMM-Zeitschrift fur Angewandte Mathematik und Mechanik)
For any $U_0 \in \mathcal{H}$, let $U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L,t), \xi_t(L,t), s_t(L,t))^\top$ the solution of the system (6). Then, there exists a positive constant C such that the following estimate holds

$$\|U_0\|^2_{\mathcal{H}} \leq C \int_0^T \left(|w(L,t)|^2 \, dt + |\xi(L,t)|^2 + |s(L,t)|^2 \right) dt. \tag{27}$$

Sketch of proof

In order to proof the inequality (27), we consider the problem

$$\begin{cases} \tilde{U}_t = \mathcal{A}\tilde{U} \\ \tilde{U}(0) = \tilde{U}_0, \end{cases} \tag{28}$$

where $\tilde{U}_0 = \mathcal{A}^{-1}U_0$. It is important to point out that the existence of $\mathcal{A}^{-1}$ is guaranteed by Lemma 2.1. Thus, from Theorem 4.1, the solution

$$\tilde{U} = (\tilde{w}, \tilde{w}_t, \tilde{\xi}, \tilde{\xi}_t, \tilde{s}, \tilde{s}_t, \tilde{w}_t(L,t), \tilde{\xi}_t(L,t), \tilde{s}_t(L,t))^\top$$

of the system (28) satisfies

$$\|\tilde{U}_0\|^2_{\mathcal{H}} \leq C \int_0^T \left(|\tilde{w}_t(L,t)|^2 \, dt + \left|\tilde{\xi}_t(L,t)\right|^2 + |\tilde{s}_t(L,t)|^2 \right) dt, \tag{29}$$

On the other hand, it is straightforward to see that $\tilde{U}_t = U$. Then, we have that

$$(\tilde{w}_t, \tilde{\xi}_t, \tilde{s}_t) = (w, \xi, s). \tag{30}$$

Finally, from (29), (30) and by Riesz representation theorem, we deduce the estimate (27). □

Due to the previous proposition, the following consequence holds.

Corollary 4.1

(Bautista, Capistrano–Filho and Límaco, submitted to ZAMM-Zeitschrift fur Angewandte Mathematik und Mechanik)

Let $W = (\chi, \chi_t, \Theta, \Theta_t, \eta, \eta_t, \chi_t(L,t), \eta_t(L,t), \Theta_t(L,t))^\top$ *is solution of the adjoint system* (14)*. Then, there exists a constant* $C > 0$*, such that the following observability inequality holds*

$$\|W(0)\|^2_{\mathcal{H}} \leq C \int_0^T \left(|\chi(L,t)|^2 \, dt + |\eta(L,t)|^2 + |\Theta(L,t)|^2 \right) dt. \tag{31}$$

As a direct consequence of the observability inequality established in Corollary 4.1, combined with the HUM method from control theory, we obtain the controllability of the laminated system with Ventcel boundary conditions.

Theorem 4.3

(Bautista, Capistrano–Filho and Límaco, submitted to ZAMM-Zeitschrift fur Angewandte Mathematik und Mechanik)
Let a time $T > 0$ *and given*

$$(w_0, w_1, \xi_0, \xi_1, s_0, s_1, , w_1(L), \xi_1(L), s_1(L))^\top \in D(\mathcal{A}),$$

one can always find a control inputs $u_i \in L^2(0,T)$, $i = 1,2,3$, *such that* (4) *admits a unique solution*

$$U = (w, w_t, \xi, \xi_t, s, s_t, w_t(L), \xi_t(L), s_t(L))^\top \in C([0,T]; D(\mathcal{A}))$$

satisfying

$$(w(T), w_t(T), \xi(T), \xi_t(T), s(T), s_t(T), w_t(L,T), \xi_t(L,T), s_t(L,T)) = \overrightarrow{0}$$

Moreover, there exists a constant $C > 0$ *such that*

$$\|(u_1, u_2, u_3)\|_{[L^2(0,T)]^3} \leq C \| (w_0, w_1, \xi_0, \xi_1, s_0, s_1, , w_1(L), \xi_1(L), s_1(L))^\top \|_{\mathcal{H}}.$$

Proof

In fact, let us consider the homogeneous system (6) with initial data $U_0 \in H'$, and let U be its corresponding solution. To evaluate this initial state in terms of boundary observables, we introduce the functional $\mathcal{J} : \mathcal{H} \to \mathbb{R}$, given by

$$\mathcal{J}(W_0) = \frac{1}{2} \int_0^T \left(|\chi(L,t)|^2 \, dt + |\eta(L,t)|^2 + |\Theta(L,t)|^2 \right) dt + \langle W_0, U_0 \rangle_{\mathcal{H} \times \mathcal{H}'} ,$$

where $W = (\chi, \chi_t, \eta, \eta_t, \Theta, \Theta_t, \chi_t(L,t), \eta_t(L,t), \Theta_t(L,t))^\top$ is solution of the adjoint system.

Note that $\mathcal{J}$ is continuous and convex. Moreover, the coercivity of functional $\mathcal{J}$ follows immediately from the observability inequality (31). Thus, thanks to [2, Corollary 3.23], we have that $\mathcal{J}$ has a minimizer. Since $\mathcal{J}$ is strictly convex, it follows that the minimizer is unique. From these facts, defining $u_1 = \chi(L,t)$, $u_2 = \eta(L,t)$ and $u_3 = \Theta(L,t)$ the null controllability holds true. □

Further comments

The main challenge stems from the need to control a system of three beam equations using three boundary inputs. As this is the first study tackling this specific problem, our results pave the way for further investigations in both the control and stabilization of such systems. Let us point out some of them.

- *Less controls*: A natural question arising from our analysis is whether it is possible to reduce the number of boundary controls. We believe that our current approach is optimal in the sense that it achieves controllability of the three equations using three boundary inputs. Nonetheless, we conjecture that, by applying Carleman estimates to the linear operator defined in (7), it may be feasible to eliminate one of the boundary controls. However, this remains an open problem at present.
- *Rapid stabilization*: It is well known in control theory that

  $$\text{observability} \Longleftrightarrow \text{controllability} \quad \text{and} \quad \text{observability} \Longrightarrow \text{rapid stabilizability}$$

  We believe that an application of an Ingham-type theorem (cf. [5]) to the system (4) could lead to a sharper observability inequality, which in turn would support the derivation of faster stabilization results. This remains another important issue to be explored.

Funding

- Bautista was supported by FAPERJ under the program PDS-2024, grant number SEI-0260003/019497/2024.
- Límaco was partially supported by CNPq grant number 310860/2023-7 (Brazil).
- Capistrano–Filho was partially supported by CAPES-COFECUB program grant number 88887.879175/2023-00, CNPq grant numbers 421573/2023-6 and 307808/2021-1, and Propesqi (UFPE).

Bautista, G.J., & Pazoto, A.F. *On the controllability of a Boussinesq system for two-way propagation of dispersive waves*, J. Evol. Equ., **20** (2020), 607–630.

Brezis, H., *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer New York, (2010), XIV, 600.

Dolecki, S., Russell, D. L., *A general theory of observation and control*, SIAM J. Control Optim., 15 (1977), 185–220.

S. W. Hansen and R. Spies, *Structural damping in a laminated beams due to interfacial slip,* J. Sound Vibration, **204** (1997), 183–202.

Ingham, A. E., *Some trigonometrical inequalities with applications to the theory of series,* Math. Z. 41 (1936), 367–369.

Komornik V., Loreti, P., *Fourier Series in Control Theory*, Springer Monographs in Mathematics. (2005).

Lions, J. L., *Exact controllability, stabilization and perturbations for distributed systems*, SIAM Rev., 30(1) (1988), 1–68.

S. Micu, *On the controllability of the linearized Benjamin-Bona-Mahony equation*, SIAM J. Cont. Optim **39** (2001), 1677–1696.

Pazy A., *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer, New York. 1983.

Simon, J., *Compact sets in the space $L^p(0, T; B)$*, Ann. Mat. Pura Appl. (IV), 146 (1987), 65–96.

Timoshenko, S.,*On the correction for shear of the differential equation for transverse vibrations of prismatic bars,* Philosophical Magazine, **41** (1921), 744–746.

Venttsel, A. D., *Semigroups of operators corresponding to a generalized differential operator of second order*, Dokl. Akad. Nauk SSSR, 111(2) (1956), 269–272.

Venttsel, A. D., *On boundary conditions for multidimensional diffusion processes*, Theory Probab. Appl., 4 (1959), 164–177.

Zuazua, E., *Exponential decay for the semilinear wave equation with locally distributed damping*, Comm. Partial Differential Equations, 15(2) (1990), 205–235.

Gracias por su atención.

Obrigado pela sua atenção.

Thanks for your attention.