Local null controllability for a parabolic equation with local and nonlocal nonlinearities in moving domains

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Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$ be a nonempty bounded connected open set, with boundary $\Gamma = \partial\Omega$ of class C^2. For $T > 0$, we represent by Q the cylinder $Q := \Omega \times (0, T)$ of $\mathbb{R}^{n+1}$, with lateral boundary $\Sigma := \Gamma \times (0, T)$.

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In this talk, we will investigate the null controllability properties of the nonlinear parabolic problem in a noncylindrical space-time domain

$$\begin{cases} u_t - a\left(\int_{\Omega_t} u \, dx, \int_{\Omega_t} \nabla u \, dx\right)\Delta u + f(u) = h\tilde{1}_{\widehat{\omega}}(1+u) & \text{in } \widehat{Q}, \\ u = 0 & \text{on } \widehat{\Sigma}, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

The noncylindrical domain $\widehat{Q}$ and its lateral boundary $\widehat{\Sigma}$ are defined by

$$\widehat{Q} := \bigcup_{0 \leqslant t \leqslant T} \{\Omega_t \times \{t\}\} \quad \text{and} \quad \widehat{\Sigma} := \bigcup_{0 \leqslant t \leqslant T} \{\Gamma_t \times \{t\}\}.$$

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Let us consider a family of functions $\{\tau_t\}_{0 \leqslant t \leqslant T}$, where for each t, τ_t is a deformation of Ω into an open bounded set Ω_t of $\mathbb{R}^n$ defined by

$$\Omega_t = \{x \in \mathbb{R}^n ; \, x = \tau_t(y) \text{ for } y \in \Omega\}.$$

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For $t = 0$, we identify Ω_0 with Ω so that τ_0 is the identity mapping. We will develop this talk under the following assumptions on the function τ_t:

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Thus we have a natural diffeomorphism $\phi : Q \to \widehat{Q}$ defined by

$$\begin{array}{rccc} \Phi : & Q & \longrightarrow & \widehat{Q} \\ & (y, t) & \mapsto & \Phi(y, t) := (x, t)\,, \end{array}$$

where,

$$(x, t) = (\tau_t(y), t) := (\tau(y, t), t)\,.$$

In (1), $u = u(x,t)$ denotes the state, u_0 is the initial data and $h = h(x,t)$ is the control which acts on the system through an arbitrarily small open set $\widehat{\omega}$, where $\widehat{\omega}$ is the image by τ_t of a non-empty open subset ω of Ω. We consider the following assumptions on a and f:

A.3 $a : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a C^1 function that possess bounded derivatives and satisfies

$$0 < a_0 < a(r,s) < a_1, \quad \forall (r,s) \in \mathbb{R} \times \mathbb{R}^n,$$

A.4 $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function, with bounded derivatives, such that $f(0) = 0$.

Definition 1.1

The system (1) is to be said null controllable at time T *if, for any* $u_0 \in H_0^1(\Omega)$*, there exist controls* $h \in L^2(\widehat{\omega} \times (0,T))$ *such that the associated solution to (1) satisfy*

$$u(\cdot, T) = 0 \quad in \quad \Omega_t. \tag{2}$$

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Our main result is the following:

Theorem 1.2

Under the previous assumptions on a and f, the nonlinear system (1) is locally null-controllable at any time $T > 0$. In other words, there exists $\varepsilon > 0$ such that, whenever $u_0 \in H_0^1(\Omega)$ and

$$\|u_0\|_{H_0^1(\Omega)} \leqslant \varepsilon,$$

there exist at least a control $h \in L^2(\widehat{\omega} \times (0, T))$ and associated state u satisfying (2).

Reduction to a fixed cylindrical domain

We first denote by $\psi_t(x)$ the inverse map of τ_t, that is, $\psi_t = {\tau_t}^{-1}$. We shall use the notation $\psi(x,t) = \psi_t(x)$. Thus the state on Q is defined by

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$$v(y,t) = u(\tau_t(y),t) = u(\tau(y,t),t) \quad \text{for all} \quad y \in \Omega. \tag{3}$$

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Equivalently in $\widehat{Q}$ we have

$$u(x,t) = v({\tau_t}^{-1}(x),t) = v(\psi(x,t),t) \text{ for all } x \in \Omega_t. \qquad (4)$$

Therefore, the initial-boundary value problem (1) is equivalent to:

$$\begin{cases} v_t + a\left(\int_\Omega v\,|J|\,dy, \int_\Omega (\nabla v \cdot \nabla \psi)\,|J|\,dy\right) A(t)v + \vec{b}\cdot\nabla v + f(v) \\ \qquad = \tilde{h}\tilde{1}_\omega(1+v) & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(y,0) = v_0(y) & \text{in } \Omega, \end{cases} \tag{5}$$

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$$
\left|\begin{array}{l}
A(t)v(y,t) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(\alpha_{ij}(y,t)\frac{\partial v}{\partial y_j}\right), \\
\alpha_{ij} = \frac{\partial \psi_i}{\partial x_k}(\tau_t(y),t)\,\frac{\partial \psi_j}{\partial x_k}(\tau_t(y),t), \\
\vec{b}(y,t) = (b_j(y,t))_{1\leqslant j\leqslant n}, \\
b_j(y,t) = \frac{\partial \psi_j}{\partial t}(\tau_t(y),t) + \sum_{i,j=1}^{n} \frac{\partial \alpha_{ij}}{\partial y_i}(y,t) - \Delta_x \psi_j(\tau_t(y),t), \\
\tilde{h}(y,t) = h(\tau_t(y),t), \;\; \tilde{1}_{\omega}(y,t) = \tilde{1}_{\widehat{\omega}}(\tau_t(y),t) \\
v_0(y) = u_0(\tau_0(y)), \\
J(y,t) \text{ is the Jacobian of the transformation } \Omega_t = \tau_t(\Omega)\,.
\end{array}\right.
$$

The system (5) is a variable coefficient parabolic equation in the cylindrical domain Q. From the technical point of view, a new problem arises because the state equation (5) contains a uniformly coercive operator $A(t)$.

The operator $A(t)$ is associated with the following bilinear form:

$$\alpha(v, u) = \sum_{i,j=1}^{n} \int_{\Omega} \alpha_{ij}(y, t) \frac{\partial v}{\partial y_j} \frac{\partial u}{\partial y_i} \, dy \, , \ \ \forall \, v, u \in H_0^1(\Omega).$$

Analysis of the controllability of the linearized system

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$$\begin{cases} v_t + a(0,\vec{0})A(t)v + \vec{b}\cdot\nabla v + cv = \tilde{h}\tilde{1}_\omega + k & \text{in} \quad Q, \\ v = 0 & \text{on} \quad \Sigma, \\ v(y,0) = v_0(y) & \text{in} \quad \Omega, \end{cases} \tag{6}$$

where the coefficient c is obtained from derivative of f at 0 and $k \in L^2(Q)$.

As usual, the controllability of (6) is closely related to the properties of the solutions to the associated adjoint states. In this case, the adjoint of (6) is given by

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$$\begin{cases} -\varphi_t + a(0,\vec{0})A^*(t)\varphi - \nabla\cdot(\vec{b}\varphi) + c\varphi = F & \text{in} \quad Q, \\ \varphi = 0 & \text{on} \quad \Sigma, \\ \varphi(T) = \varphi_T & \text{in} \quad \Omega, \end{cases} \tag{7}$$

where $A^*(t)$ is the formal adjoint of the operator $A(t)$, $F \in L^2(Q)$ and $\varphi_T \in L^2(\Omega)$.

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where $A^*(t)$ is the formal adjoint of the operator $A(t)$, $F \in L^2(Q)$ and $\varphi_T \in L^2(\Omega)$.

Next we sketch the points used in the proof of the null controllability of the system (6) using suitable Carleman estimates. First, we use a global Carleman inequality satisfied by the solutions to (7). Second, this inequality allows us to establish an observability estimate. Third, we prove a new Carleman inequality with weights that do not vanish at $t = 0$. Finally, we prove the null controllability of (6) by using the new Carleman estimate.

In this approach, the following technical result due to Fursikov and Imanuvilov, is fundamental.

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Lemma 2.1

Let $\omega_1 \subset\subset \omega$ a non-empty open subset. Then, there exists a function $\beta_0 \in C^2(\overline{\Omega})$ satisfying:

- $\beta_0(y) > 0 \;\; \forall y \in \Omega,$
- $\beta_0 = 0 \;\; \forall y \in \partial\Omega,$
- $|\nabla \beta_0(y)| > 0 \;\; \forall y \in \overline{\Omega} - \omega_1.$

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Let us introduce the weight functions

$$\alpha(y,t) := \frac{\alpha_1(y)}{\beta(t)}, \quad \phi(y,t) := \frac{e^{\lambda\beta_0(y)}}{\beta(t)} \quad \text{with}$$
$$\alpha_1(y) := e^{R\lambda} - e^{\lambda\beta_0(y)}, \quad \beta(t) := t(T-t), \, 0 < t < T, \tag{8}$$

where, $R > \|\beta_0\|_{L^\infty(\Omega)} + \ln 6$ and $\lambda > 0$.

Also, let us set

$$\alpha^*(t) := \frac{\max_{y\in\overline{\Omega}} \alpha_1(y)}{\beta(t)}, \quad \widehat{\alpha}(t) := \frac{\min_{y\in\overline{\Omega}} \alpha_1(y)}{\beta(t)}$$
$$\phi^*(t) := \max_{y\in\overline{\Omega}} \phi(y,t), \quad \widehat{\phi}(t) := \min_{y\in\overline{\Omega}} \phi(y,t)\,,$$
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The following global Carleman estimate hold for the solution to (7):

Proposition 2.2

There exist positive constants λ_0, s_0 and C_0 such that, for any $\lambda \geqslant \lambda_0$, $s \geqslant s_0$ and any $\varphi_T \in L^2(\Omega)$ and $F \in L^2(Q)$, the associated solution to (7) satisfies

$$\iint_Q e^{-2s\alpha}\Big[(s\phi)^{-1}\Big(|\varphi_t|^2+\sum_{i,j=1}^{n}\Big|\frac{\partial^2\varphi}{\partial y_j\partial y_i}\Big|^2\Big)+\lambda^2(s\phi)|\nabla\varphi|^2 +\lambda^4(s\phi)^3|\varphi|^2\Big]dydt \leqslant C_0\Big(\iint_Q e^{-2s\alpha}|F|^2\,dydt+\iint_{\omega_1\times(0,T)} e^{-2s\overline{\alpha}}\lambda^4(s\phi^*)^3|\varphi|^2\,dydt\Big). \tag{9}$$

Furthermore, C_0 and λ_0 only depend on Ω and ω and s_0 can be chosen of the form $s_0 = C(\Omega,\omega)(T+T^2)$.

An important consequence of Proposition 2.2 is the following observability inequality:

Corollary 1

Suppose that $F = 0$ in (7). Then, there exists a positive constant C depending on $T, s$and λ, such that, for any solution to (6), one has:

$$\|\varphi(0)\|_{L^2(\Omega)} \leqslant C \iint_{\omega_1 \times (0,T)} e^{-2s\overline{\alpha}} (\phi^*)^3 |\varphi|^2 \, dydt,$$

where s and λ are taken as in Proposition 2.2.

We will also need some Carleman inequalities for the solution to (7) with suitable weights that do not vanish at $t = 0$. To this end, let m be a function satisfying

$$m \in C^\infty([0, T]), \;\; m(t) \geqslant \frac{T^2}{4} \text{ in } [0, T/2], \;\; m(t) = t(T-t) \text{ in } [T/2, T].$$

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We consider

$$\begin{gathered}
\theta(y,t) := \tfrac{e^{\lambda\beta_0(y)}}{m(t)}, \quad A(y,t) := \tfrac{\alpha_1(y)}{m(t)}, \\
A^* := \max_{y\in\overline{\Omega}} \alpha_1(y), \quad \widehat{A} := \min_{y\in\overline{\Omega}} \alpha_1(y)\,, \\
\theta^*(t) := \max_{y\in\overline{\Omega}} \theta(y,t), \quad \widehat{\theta}(t) := \min_{y\in\overline{\Omega}} \theta(y,t)\,, \\
\overline{A} := 2\widehat{A} - A^*.
\end{gathered}$$

Next, we will use the following notation:

$$I(\varphi) := \iint_Q e^{-2sA} \Big[(s\theta)^{-1} \Big(|\varphi_t|^2 + \sum_{i,j=1}^{n} \Big| \frac{\partial^2 \varphi}{\partial y_j \partial y_i} \Big|^2 \Big) + \lambda^2 (s\theta) |\nabla \varphi|^2 + \lambda^4 (s\theta)^3 |\varphi|^2 \Big] \, dydt.$$

One has the following:

Proposition 2.3

Under the assumptions of Proposition 2.2, there exist positive constants λ_2, s_2 such that, for any $\lambda \geqslant \lambda_2$ and $s \geqslant s_2$, there exists $C_2(s,\lambda)$ with the following property: for any $\varphi_T \in L^2(\Omega)$ the associated solution to (7) satisfies

$$I(\varphi) \leqslant C_2\Big(\iint_Q e^{-2sA}|F|^2\ dydt + \iint_{\omega_1\times(0,T)} e^{(-2s\overline{A})/m}(\theta^*)^3|\varphi|^2\ dydt\Big), \tag{10}$$

where λ_2, s_2 only depend on $\Omega, \omega, T, a(0,\vec{0}), |b|$ and $C_2(s,\lambda)$ only depend on these data s and λ.

In the sequel, we fix $\lambda = \lambda_2,\ s = s_2$ and we set

$$\rho := e^{sA},\ \rho_0 := e^{sA}\theta^{-3/2},\ \rho_* := e^{(s\overline{A})/m}(\theta^*)^{-3/2},$$
$$\widehat{\rho} := e^{(sA)/2m}e^{(s\overline{A})/2m}(\theta^*)^{-3/2},\ \overline{\rho} := e^{s(4\widehat{A}-3A^*)/m}(\theta^*)^{-5}.$$

Then, we deduce from (10) that the solution to (7) satisfies:

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Thanks to Proposition 2.3, we will be able to prove the null controllability to (6) for the right hand side k that decay fast to zero as $t \to T$. Indeed, one has the following:

Theorem 2.4

Assume that the function k *satisfy*

$$\iint_Q \rho_0^2 |k|^2 \; dydt < +\infty.$$

Then (6) is null controllable. More precisely, for any $v_0 \in L^2(\Omega)$*, there exist controls* $\tilde{h} \in L^2(\omega \times (0,T))$ *such that the state-control* $(v, \tilde{h})$ *satisfies*

$$\iint_{\omega\times(0,T)} \rho_*^2 |\tilde{h}|^2 dydt \leqslant C\left(\|v_0\|^2_{L^2(\Omega)} + \iint_Q \rho_0^2 |k|^2 dydt\right), \qquad (11)$$

$$\iint_Q \rho_0^2 |v|^2 dydt \leqslant C\left(\|v_0\|^2_{L^2(\Omega)} + \iint_Q \rho_0^2 |k|^2 dydt\right),$$

whence, in particular, $v(y,T) = 0$ *in* Ω*.*

Regularity of control

The next result to be stated, establishes regularity of control. This is a novelty in the present work. Such regularity is important to analyze the estimates involving multiplicative control.

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Proposition 2.5

Let $\overline{\rho} = e^{s(4\widehat{A}-3A^*)/m}(\theta^*)^{-5}$. *Then one has*

$$\overline{\rho}\tilde{h} \in V := L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)) \quad (12)$$

and

$$\|\overline{\rho}\tilde{h}\|_V \leqslant C(\|v_0\|_{L^2(\Omega)} + \|\rho_0 k\|_{L^2(Q)}) \quad (13)$$

The local null controllabilty of the nonlinear problem

The aim of this section is to prove Theorem 1.2. For this, we will use the Liusternik's Inverse Function Theorem.

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Let us start by defining the following function spaces:

$$Y := \Big\{(v,\tilde{h}) : \tilde{h} \in L^2(\omega\times(0,T)),\ \iint_{\omega\times(0,T)} \rho_*^2|\tilde{h}|^2\ dydt < +\infty,$$

$$v,\ \partial_i v,\ v_t + a(0,\vec{0})A(t)v \in L^2(Q)\,,\ \iint_Q \rho_0^2|v|^2\ dydt < +\infty,$$

$$\iint_Q \rho_0^2|v_t + a(0,\vec{0})A(t)v + \vec{b}\cdot\nabla v - \tilde{h}1_\omega|^2\,dydt < +\infty$$

$$v(\cdot,0)\in H_0^1(\Omega)\Big\}\,,$$

$$G := \Big\{k \in L^2(Q) : \iint_Q \rho_0^2|k|^2\ dydt < +\infty\Big\}$$

and

$$Z := G\times H_0^1(\Omega).$$

We consider these spaces with the following Hilbertian norms:

$$\|(v, \tilde{h})\|_Y^2 := \iint_Q \rho_0^2 |v|^2 \; dydt + \iint_{\omega\times(0,T)} \rho_*^2 |\tilde{h}|^2 \; dydt$$
$$+ \iint_Q \rho_0^2 \Big(|v_t + a(0,\vec{0})A(t)v + \vec{b}\cdot\nabla v - \tilde{h}1_\omega|^2 \Big) \; dydt$$
$$+ \|v(\cdot,0)\|_{H_0^1(\Omega)}^2,$$

$$\|k\|_G^2 := \iint_Q \rho_0^2 |k|^2 \; dydt$$

and

$$\|(k, v(\cdot,0))\|_Z^2 := \|k\|_G^2 + \|v(\cdot,0)\|_{H_0^1(\Omega)}^2.$$

Let us consider the mapping $H : Y \to Z$ with

$$H(v,\tilde{h}) = (H_1, H_2)(v,\tilde{h}), \tag{14}$$

$$\begin{aligned} H_1(v,\tilde{h}) =& v_t + a\left(\int_\Omega v\,|J|\;dy, \int_\Omega (\nabla v\cdot\nabla\psi)\,|J|\;dy\right)A(t)v + \\ &+ \vec{b}\cdot\nabla v + f(v) - \tilde{h}\tilde{1}_\omega(1+v)\,, \end{aligned}$$

$$H_2(v,\tilde{h}) := v(\cdot,0).$$

We will prove that there exists $\varepsilon > 0$ such that, if $(k, v_0) \in Z$ and $\|(k,v_0)\|_Z \leqslant \varepsilon$, then the equation

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possesses at least one solution.

In particular, this show that (1) is locally null controllable and, furthermore, the state-control pair can be chosen in Y.

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Let Y and Z be Banach spaces and let $H : B_r(0) \subset Y \to Z$ be a C^1 mapping. Let us assume that the derivative $H'(0) : Y \to Z$ is onto and let us set $\xi_0 = H(0)$. Then there exist $\varepsilon > 0$, a mapping $W : B_\varepsilon(\xi_0) \subset Z \to Y$ and a constant $K > 0$ satisfying

$$\begin{cases} W(z) \in B_r(0) \text{ and } H(W(z)) = z, \ \forall z \in B_\varepsilon(\xi_0), \\ \|W(z)\|_Y \leqslant K\|z - H(0)\|_Z, \ \forall z \in B_\varepsilon(\xi_0). \end{cases}$$

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To show that Theorem 3.1 can be applied in this setting, we will use several lemmas.

Lemma 2

Let $H : Y \to Z$ be the mapping defined by (14). Then H is well defined and continuous.

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Lemma 3

The mapping $H : Y \to Z$ is continuously differentiable.

Lemma 2

Let $H : Y \to Z$ *be the mapping defined by (14). Then* H *is well defined and continuous.*

Lemma 3

The mapping $H : Y \to Z$ *is continuously differentiable.*

Lemma 4

Let H *be the mapping defined by (14). Then* $H'(0,0) : Y \to Z$ *is onto.*

In view of Lemmas 2, 3 and 4, we can apply Liusternik's Theorem to the mapping $H : Y \to Z$ and (5) is locally null controllable, with $(v, \tilde{h}) \in Y$.

In view of Lemmas 2, 3 and 4, we can apply Liusternik's Theorem to the mapping $H : Y \to Z$ and (5) is locally null controllable, with $(v, \tilde{h}) \in Y$.

Consequently, by using the diffeomorphism $(y, t) \to (x, t)$ from Q to $\widehat{Q}$, we obtain that (1) is null controllable.

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Thank you!